

Prof. Hadley

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EDITED BY

J. D. RUNKLE, A.M., A.A.S.

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A complete Index to the work is in press, and will be issued shortly.

THE MATHEMATICAL MONTHLY.

The excellent notices which the Monthly continues to receive from both sides of the Atlantic must be a source of great encouragement to all its friends. The *Lady's and Gentlemen's Diary* for 1861, edited by the eminent mathematician, PROF. W. S. B. WOOLHOUSE, contains the following:—

"This American periodical, which has now completed its second volume, continues to be most ably conducted, and is well supplied by talented correspondents. Besides numerous mathematical questions and solutions, it comprises original papers of considerable variety and sterling interest, and its publication must contribute largely to the diffusion and advancement of mathematical science."

A recent number of the *Boston Daily Transcript* gives the following item:—

"AMERICAN LITERATURE ABROAD.—Fourteen columns of the last *Athenæum* are devoted to two American authors,—Motley and Emerson. That paper, which rarely commends anything from this side of the Atlantic, gives 'The Mathematical Monthly' the following first-rate notice. That magazine is published at Cambridge, and edited by Mr. J. D. Runkle:—

"This is a collection of simple things for comparative elementary students. Our English mathematical journals are very lofty, and the beginner has no part in them. This American journal is full of pleasing elementary matter, and contains some things worthy the attention of the finished mathematician. We should like to see the same sort of thing in England; but where are we to find the editor? He must be of high knowledge and elementary turn combined, with firmness enough to keep his journal down to the intelligible point by refusal of contributions of too lofty a character, and skill enough to keep it up to the interesting point by selecting from the loads of lower learning with which he would be inundated."

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The Mathematical Monthly.

In commencing our third volume, we beg to assure the friends of this Journal, that all legitimate efforts will be made to sustain it, together with the Prizes, which have proved such an excellent feature of its plan. It will be seen (see 2d page of cover) that the amount of the Prizes has been increased. But all efforts on our part must prove unavailing if the subscription list, with other sources of income, will not cover the expenses which we are incurring to make the Monthly really valuable and attractive to all interested in it. To show our friends at home what is thought of the Monthly abroad, we venture to make short extracts from letters lately received.

"The work appears to me so excellent, that I have at once ordered the series for the Library of the Observatory.

"G. B. AIRY, Ast. Royal."

"Royal Obs., Greenwich, July 5, 1860.

"I have looked through it with great interest, and especially because it shows to what an extent interest in mathematical subjects is felt through the United States. I do not believe that it would be possible in England to get up a Mathematical periodical brought out in so handsome a form and supported by so large a number of subscribers.

GEORGE SALMON."

"Trinity College, Dublin, July 12, 1860.

"DEAR SIR, — I thank you sincerely for the copy you so kindly sent me of the first volume of the Mathematical Monthly. We have not, that I am aware of, any work of a similar character in England. I feel confident it will serve a valuable purpose in stimulating and giving a healthy tone to the study of Mathematics in the higher class of schools, and judging from this volume, it will be a work of as great interest and give as much matter for thought to the tutor as to the student. Original investigations on the mere elementary branches of science are perhaps not often to be expected: those portions have now been so long and so frequently considered, and by so many different minds, that something new — a really original idea — appears only at very distant intervals. Still many interesting questions may, and, as your first volume shows, do arise even in the earliest branches of Mathematical science; whether discussions on the principles and on the best methods of placing them before the pupil, or examples and problems which in their solutions show a large amount of ingenuity and mathematical skill. The Mathematical Monthly claims a special interest from the very fact of its character rendering it useful to the many without detracting from its scientific merit.

"I remain, your obedient servant,

"St. John's College, Cambridge, July 26, 1860.

HUGH GODFREY."

We feel confident that a little effort on the part of our friends would double our subscription list. See terms to clubs and new subscribers taking Vols. I., II., III. We could then engrave some of the fine portraits of LAGRANGE, MONTE, CAUCHY, BIOT, &c., lately received; and, indeed, incur any expense to increase the value of the Monthly to all. Will our friends make up their clubs at once, and notify us, so that we can decide upon the size of our edition?

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THE MATHEMATICAL MONTHLY.

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PRIZE PROBLEMS FOR STUDENTS.

I. If a quadrilateral circumscribe a circle, its diagonals, and the lines joining the points of contact of opposite sides, meet in a point.

II. Two men, A and B , agree to perform two equal pieces of work. A begins one piece, and B the other; and after working p days they exchange pieces. A finishes his piece in q days, and then returns and helps B ; and after working together r days they finish his piece. How long would each be in doing either piece? If A is twice as good a workman as B , show that $q = p + 3r$.

III. Prove that, in any triangle,

$$(1) \quad \cot B - \cot A = \frac{a^2 - b^2}{a b \sin c}, \quad (2) \quad \frac{\sin(\frac{1}{2}A + B)}{\sin \frac{1}{2}A} = \frac{b + c}{a},$$

$$(3) \quad \text{Area} = \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C \left(\frac{a^2}{\sin A} + \frac{b^2}{\sin B} + \frac{c^2}{\sin C} \right).$$

IV. Find the number of conditions that must be satisfied by the coefficients of the general equation of the n th degree, that it may represent a series of parallel lines at equal distances from each other.

V. The population of a town at the end of any year can be found by subtracting eleven times the population at the end of the previous year from ten times the population at the end of the succeeding year. Nine years ago the population was 1210, and eleven years 1000. Show that it increases in geometrical progression.

Solutions of these problems must be received by August 1, 1861.

TO CHANGE A SERIES INTO A CONTINUED FRACTION.

By A. HALL, Cambridge, Mass.

GAUSS has shown, in his "Disquisitiones Generales circa seriem infinitam, etc.," that nearly all the series which commonly occur in algebraic analysis are contained as special cases in the following one:—

$$(1) \ 1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+2)}x^3 + \text{etc.}$$

The quantities α, β, γ, x are called the *elements* of this series, and are distinguished in their order as the first, second, third, and fourth elements.

The following are some of the principal properties of this series.

Since the first and second elements enter the series symmetrically, we can change places with them, and the value of the series will not be altered. Denoting, therefore, the series by $F(\alpha, \beta, \gamma, x)$, we shall have $F(\alpha, \beta, \gamma, x) = F(\beta, \alpha, \gamma, x)$.

By giving to the elements α, β, γ definite values, the series becomes a function of a single variable x , and is evidently broken off after the $1-\alpha$ th or the $1-\beta$ th term, if $\alpha-1$ or $\beta-1$ is a negative whole number; but in all other cases the series extends to infinity. In the first case, the series becomes a rational algebraic function; but in the last, it is commonly a transcendental function. The third element, γ , can be neither a negative whole number nor $= 0$, lest we fall upon terms infinitely great.

The coefficients of the powers x^m, x^{m+1} are as $1 + \frac{r+1}{m} + \frac{r}{m^2}$ is to $1 + \frac{\alpha+\beta}{m} + \frac{\alpha\beta}{m^2}$, and therefore approach a ratio of equality as m is increased. Hence the convergency of the series depends on the value given to the fourth element, x , and the series will always converge if x has a real value, positive or negative, and less than unity.

By giving to the elements proper values, it is easy to represent some of the more common series. Thus,

$$(a) \quad (t + u)^n = t^n \cdot F\left(-n, \beta, \beta, -\frac{u}{t}\right),$$

when the element β is still arbitrary.

$$(b) \quad \log(1 + t) = t \cdot F(1, 1, 2, -t),$$

$$(c) \quad \log \frac{1+t}{1-t} = 2t \cdot F\left(\frac{1}{2}, 1, \frac{3}{2}, t^2\right),$$

$$(d) \quad e^t = F\left(1, k, 1, \frac{t}{k}\right) = 1 + t \cdot F\left(1, k, 2, \frac{t}{k}\right) = 1 + t + \frac{1}{2}t^2 \cdot F\left(1, k, 3, \frac{t}{k}\right), \text{etc.},$$

when e denotes the base of hyperbolic logarithms and k is a number infinitely great. Also,

$$(e) \quad \arcsin t = \sin t \cdot F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \sin^2 t\right) = \sin t \cos t \cdot F\left(1, 1, \frac{3}{2}, \sin^2 t\right).$$

The series (1) possesses the important property, that, if we subtract it from a similar series, the remainder is a series of the same form. Writing, in (1), $\beta + 1$ and $\gamma + 1$ for β and γ , we shall have

$$F(\alpha, \beta + 1, \gamma + 1, x) = 1 + \frac{\alpha(\beta + 1)}{1 \cdot (\gamma + 1)}x + \frac{\alpha(\alpha + 1)(\beta + 1)(\beta + 2)}{1 \cdot 2 \cdot (\gamma + 1)(\gamma + 2)}x^2 + \text{etc.},$$

and, subtracting (1),

$$F(\alpha, \beta + 1, \gamma + 1, x) - F(\alpha, \beta, \gamma, x) = \frac{\alpha(\gamma - \beta)x}{\gamma(\gamma + 1)} \cdot \left[1 + \frac{(\alpha + 1)(\beta + 1)}{1 \cdot (\gamma + 2)}x + \frac{(\alpha + 1)(\alpha + 2)(\beta + 1)(\beta + 2)}{1 \cdot 2 \cdot (\gamma + 2)(\gamma + 3)}x^2 + \text{etc.}, \right]$$

or,

$$(2) \quad F(\alpha, \beta + 1, \gamma + 1, x) - F(\alpha, \beta, \gamma, x) = \frac{\alpha(\gamma - \beta)x}{\gamma(\gamma + 1)} \cdot F(\alpha - 1, \beta - 1, \gamma - 2, x).$$

And in the same way

$$(3) \quad F(\alpha - 1, \beta + 1, \gamma, x) - F(\alpha, \beta, \gamma, x) = \frac{(\alpha - \beta - 1)x}{\gamma} \cdot F(\alpha, \beta + 1, \gamma + 1, x).$$

Two more similar equations can be immediately obtained by interchanging α and β ; and, by the same process by which equations (2)

and (3) have been obtained, we may find many other such equations. This last property can be used to transform the series (1) into a continued fraction.

Designating $\frac{F(\alpha, \beta+1, \gamma+1, x)}{F(\alpha, \beta, \gamma, x)}$ by $G(\alpha, \beta, \gamma, x)$, we have, by interchanging α and β ,

$$\frac{F(\beta, \alpha+1, \gamma+1, x)}{F(\beta, \alpha, \gamma, x)} = G(\beta, \alpha, \gamma, x);$$

and since $F(\alpha, \beta, \gamma, x) = F(\beta, \alpha, \gamma, x)$, we have

$$F(\beta, \alpha+1, \gamma, x) = F(\alpha+1, \beta, \gamma+1, x),$$

and therefore

$$(4) \quad \frac{F(\alpha+1, \beta, \gamma+1, x)}{F(\alpha, \beta, \gamma, x)} = G(\beta, \alpha, \gamma, x).$$

Dividing equation (2) by $F(\alpha, \beta+1, \gamma+1, x)$, we have, by equation (4),

$$1 - \frac{1}{G(\alpha, \beta, \gamma, x)} = \frac{\alpha(\gamma-\beta)x}{\gamma(\gamma+1)} \cdot G(\beta+1, \alpha, \gamma+1, x),$$

where, in $G(\beta+1, \alpha, \gamma+1, x)$, $\beta+1$ and $\gamma+1$ are written for β and γ . Hence,

$$G(\alpha, \beta, \gamma, x) = \frac{1}{1 - \frac{\alpha(\gamma-\beta)x}{\gamma(\gamma+1)} \cdot G(\beta+1, \alpha, \gamma+1, x)}.$$

Again, by interchanging α and β , and writing $\beta+1$ and $\gamma+1$ for β and γ , we get

$$G(\beta+1, \alpha, \gamma+2, x) = \frac{1}{1 - \frac{(\beta+1)(\gamma+1-\alpha)x}{(\gamma+1)(\gamma+2)} \cdot G(\alpha+1, \beta+1, \gamma+2, x)},$$

$$G(\alpha+1, \beta+1, \gamma+2, x) = \frac{1}{1 - \frac{(\alpha+1)(\gamma+1-\beta)x}{(\gamma+2)(\gamma+3)} \cdot G(\beta+2, \alpha+1, \gamma+3, x)},$$

etc.

Whence, by repeated substitutions of the function G , there will result for $G(\alpha, \beta, \gamma, x)$ the continued fraction,

$$(5) \quad \frac{F(\alpha, \beta+1, \gamma+1, x)}{F(\alpha, \beta, \gamma, x)} = \frac{1}{1 - \frac{ax}{1 - \frac{bx}{1 - \frac{cx}{1 - \frac{dx}{1 - \text{etc.}}}}}}$$

where

$$\begin{aligned} a &= \frac{\alpha(\gamma - \beta)}{\gamma(\gamma + 1)}, & b &= \frac{(\beta + 1)(\gamma + 1 - \alpha)}{(\gamma + 1)(\gamma + 2)}, \\ c &= \frac{(\alpha + 1)(\gamma + 1 - \beta)}{(\gamma + 2)(\gamma + 3)}, & d &= \frac{(\beta + 2)(\gamma + 2 - \alpha)}{(\gamma + 3)(\gamma + 4)}, \\ e &= \frac{(\alpha + 2)(\gamma + 2 - \beta)}{(\gamma + 4)(\gamma + 5)}, & f &= \frac{(\beta + 3)(\gamma + 3 - \alpha)}{(\gamma + 5)(\gamma + 6)}, \\ &\text{etc.,} & &\text{etc.,} \end{aligned}$$

in which the law of progression is obvious. It is evident that this continued fraction will stop if either of the numbers $\alpha, \beta, \gamma - \alpha, \gamma - \beta$ is a negative whole number, but otherwise it will extend to infinity.

If in formula (5) we put $\beta = 0$, whence $F(\alpha, \beta, \gamma, x) = 1$, and write $\gamma - 1$ for γ , we shall have

$$\begin{aligned} (6) \quad F(\alpha, 1, \gamma, x) &= 1 + \frac{\alpha}{\gamma}x + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}x^3 + \text{etc.} \\ &= \frac{1}{1 - \frac{ax}{1 - \frac{bx}{1 - \frac{cx}{1 - \frac{dx}{1 - \text{etc.}}}}}} \end{aligned}$$

where we now have

$$\begin{aligned} a &= \frac{\alpha}{\gamma}, & b &= \frac{\gamma - \alpha}{\gamma(\gamma + 1)}, \\ c &= \frac{(\alpha + 1)\gamma}{(\gamma + 1)(\gamma + 2)}, & d &= \frac{2(\gamma + 1 - \alpha)}{(\gamma + 2)(\gamma + 3)}, \\ e &= \frac{(\alpha + 2)(\gamma + 1)}{(\gamma + 3)(\gamma + 4)}, & f &= \frac{3(\gamma + 2 - \alpha)}{(\gamma + 4)(\gamma + 5)}, \\ &\text{etc.} & & \end{aligned}$$

By means of equation (6) we can express the equations (a), (b), (c), etc., in continued fractions. Putting $t = 1$, $\beta = 1$, we have, for the binomial,

$$(1+u)^n = \frac{1}{1 - \frac{n u}{1 + \frac{\frac{n+1}{2} u}{1 - \frac{\frac{n-1}{2 \cdot 3} u}{1 + \frac{\frac{2(n+2)}{3 \cdot 4} u}{1 - \frac{2(n-2)}{4 \cdot 5} u, \text{ etc.}}}}}}$$

Equation (c) gives

$$\log \frac{1+t}{1-t} = \frac{2t}{1 - \frac{1}{3} t^2} \div \frac{1 - \frac{2 \cdot 2}{3 \cdot 5} t^2}{1 - \frac{3 \cdot 3}{5 \cdot 7} t^2} \div \frac{1 - \frac{4 \cdot 4}{7 \cdot 9} t^2, \text{ etc.}}$$

By changing in this last equation the signs — into +, we shall have a continued fraction for arc. tan t . Again, if in the preceding fraction for the binomial $(1+u)^n$ we put $n = k$, and $u = \frac{t}{k}$, we shall have, by making k infinite, a continued fraction for the exponential e^t .

If we put $\alpha = 3$, $\gamma = \frac{5}{2}$, in equation (6), we have immediately the first continued fraction in article 90, *Theoria Motus*. For the second one of this article, let

$$Q = 1 - \frac{\frac{5 \cdot 8}{7 \cdot 9} x}{1 - \frac{1 \cdot 4}{9 \cdot 11} x, \text{ etc.,}}$$

and we have

$$x - \xi = \frac{x}{1 + \frac{2x}{35Q}} = \frac{xQ}{Q + \frac{2x}{35}},$$

and therefore

$$\xi = \frac{\frac{2}{35}x^2}{Q + \frac{2}{35}x},$$

which is the second formula, or [13]. The third is found in the following manner.

$$\begin{aligned} \text{Putting } R &= 1 - \frac{1.4}{7.9}x \\ &\quad 1 - \frac{5.8}{9.11}x \\ &\quad 1 - \frac{3.6}{11.13}x \\ &\quad 1 - \frac{7.10}{13.15}x, \text{ etc.,} \end{aligned}$$

we find, by equation (5),

$$\frac{1}{R} = G\left(\frac{1}{2}, \frac{3}{2}, \frac{7}{2}, x\right), \quad \text{and} \quad \frac{1}{Q} = G\left(\frac{5}{2}, -\frac{1}{2}, \frac{7}{2}, x\right).$$

Hence,

$$R \cdot F\left(\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, x\right) = F\left(\frac{1}{2}, \frac{3}{2}, \frac{7}{2}, x\right),$$

$$Q \cdot F\left(\frac{5}{2}, \frac{1}{2}, \frac{9}{2}, x\right) = F\left(\frac{5}{2}, -\frac{1}{2}, \frac{7}{2}, x\right);$$

or, by interchanging the first and second elements in this last equation,

$$Q \cdot F\left(\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, x\right) = F\left(-\frac{1}{2}, \frac{5}{2}, \frac{7}{2}, x\right).$$

But by equation (3) we have

$$F\left(-\frac{1}{2}, \frac{5}{2}, \frac{7}{2}, x\right) - F\left(\frac{1}{2}, \frac{3}{2}, \frac{7}{2}, x\right) = -\frac{4}{7}x \cdot F\left(\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, x\right),$$

and therefore, substituting and dividing by $F\left(\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, x\right)$, we have

$$Q = R - \frac{4}{7}x;$$

and substituting this in the preceding value for ξ ,

$$\xi = \frac{\frac{2}{35}x^2}{R - \frac{1}{35}x},$$

which is the last formula.

The preceding method of transforming a series into a continued fraction was first given by GAUSS, in the Memoir referred to at the beginning of this article, and which is printed in the Memoirs of the Göttingen Society, 1812. See also, on the subject, CRELLE's Journal, Vol. X., where there is a very elaborate article by STERN, GERGONNE's *Annales*, Vol. IX., and a little book called *Algebraische Analysis*, by Dr. O. SCHLÖMILCH.

ON THE GEOMETRICAL CONSTRUCTION OF CERTAIN CURVES BY POINTS.

By H. A. NEWTON, Professor of Mathematics in Yale College.

[Continued from page 244.]

33. THE method given by CHASLES for constructing the cubic through nine points is such that with the straight edge alone can be constructed the point where a straight line through two points again meets the curve.* Hence any number of points in the problem of the last article can be constructed by the straight edge alone.

34. The construction of a biquadratic to pass through fourteen single points is given by DE JONQUIERES (LIOUVILLE, 2d series, Vol. I. p. 416). Hence any construction of a curve which can be reduced to the construction of such a biquadratic is possible.

35. To construct a curve of the n th order (n being not less than 5 and not greater than 15), having given $(15 - n)$ single points,

* This fact is not mentioned by CHASLES, though he must have known it.

3 double points, $(n-5)$ triple points, and 1 point of the $(n-4)$ th order.

Take the point of the $(n-4)$ th order, a triple point, and a double point as base points, and the construction depends on that of a curve of the $(n-1)$ st order, $(15-(n-1))$ single points, three double points, $(n-6)$ triple points, and one point of the $(n-5)$ th order; that is, the construction depends on that of the curve of the $(n-1)$ st order, to satisfy conditions like those of the original problem. By repeating the process, the construction depends, finally, on that of a curve of the fifth degree, having three given double points and eleven single points. Taking the three double points as base points, the construction depends on that of the bi-quadratic through fourteen points.

36. These theorems can be multiplied indefinitely. The following table is arranged to show in what cases curves of the fourth, fifth, sixth, seventh, and eighth degrees can be constructed. The asterisks indicate the constructions that are possible with the straight edge alone.

	I.	II.	III.	IV.		I.	II.	III.	IV.	V.		I.	II.	III.	IV.	V.	VI.		I.	II.	III.	IV.	V.	VI.	VII.
4	14					6	6	1	3		*	7	7	3	1		*		8	6	2	2	2		*
	11	1					11	2		1	*		11	3			*			3	3	2	2		*
	8	2					8	3		1	*		8	4			*		14	1		3	3		*
	5	3					5	4		1	*		5	5			*		11	1		3	3		*
	8		1				12			1	*		14				*		8	2		3	3		*
																			5	3		3			*
5	11	3				7	8	3	3				8	6					8		1	3			*
	8	4					5	4	3				5	1	6				5	8			1		*
	5	5					2	5	3				2	2	6				2	9					*
	2	6					8	1	4				2	3	7				8	5	1	1	1		*
	11	1					5	2	4				7	3	3	1			5	6	1	1	1		*
	8	2	1				2	3	4				4	4	3				2	7	1	1	1		*
	5	3	1				7	6		1			1	5	3				11	2	2	1	1		*
	10			1			4	7		1			10	4	1				8	3	2	1	1		*
							1	8					7	1	4	1			5	4	2	1	1		*
6	9	4	1				10	3	1	1			4	2	4	1			2	5	2				*
	6	5	1				7	4	1	1			1	3	4	1			11	1	3	1	1		*
	3	6	1				4	5	1	1			4		5	1			8	3	3		1		*
	7	7	1				1	6	1	1			9	3	1	2			5	2					*
	12	1	2				13		2	1			6	4	1	2			11	4			1		*
	9	2	2				10	1	2	1			3	5	1	2			8	5					*
	6	3	2				7	2	2	1			6	1	2				5	6					*
	3	4	2				4	3	2	1			9	1	2				16				1		*
	9		3																						*

The arrangement of the table will be evident from an example. Thus, the last line of the third column indicates that a curve of the eighth degree can be constructed, having given nine single points, one double point, two triple points, and two quadruple points.

37. Curves, *properly*, of the degree mentioned in the first column of the table below, and having the number of multiple points indicated, are impossible.

	II.	III.	IV.	V.	VI.
6	5	2			
	5		1		
7	4	2	1		
	6			1	
8	1	5	1		
	4	2	2		
	3	3		1	
	7				1

For in each case, if the largest multiple points be taken as base points, and the largest multiple points of the conjugate curve be again taken as base points, and so on, we have at last a conic with a double point, or a cubic with two double points, either of which is impossible.

38. The given points have been thus far supposed all real. But they may also be imaginary in couples.

Two imaginary points may be considered as given when they are the points of intersection of a given straight line with a conic section, of which five points are given. For these data enable us to determine with the rule and compasses the various real relations of the imaginary points.

The method of determining two imaginary points on a given line by their middle point, and the product of their distances to a given point of the line, is not suitable to constructions in which the straight edge alone is to be used. With the compasses, however, we can describe a circle that shall pass through the points thus given. Five points on this circle can be used in these constructions.

39. PROBLEM. — To construct a sixth point of a conic that shall pass through two imaginary and three real points.

Let the two imaginary points be the intersection of the straight line l with the conic $abcde$, and let the three other given points

be P , Q , and R . On \mathcal{L} take three points, m , n , and o , and construct (Art. 25) the points m' , n' , and o' , where the three conics, $abcdm$, $abcdn$, and $abcdo$, meet again the line \mathcal{L} . Draw the straight line ae , and construct on this line the three points m'' , n'' , and o'' , where the same three conics cut it.

Through P draw any straight line, and on this line construct the points m''' , n''' , and o''' , where the three conics $PQRmm'$, $PQRnn'$, $PQRoo'$ cut it. Then on this line construct a fourth point, e' , such that the anharmonic function of m''' , n''' , o''' , e' is equal to that of m'' , n'' , o'' , e . (*Géom. Sup.*, 113.) The point e' is a required point.

For the series of conics circumscribing the quadrilateral $abcd$ cuts \mathcal{L} in a series of couples of points in involution, of which the two given imaginary points form one couple, and the middle points of these couples correspond anharmonically with the series of points where the same conics cut the line ae . Moreover, the conics through PQR , and the couples of points in involution on \mathcal{L} , pass all through a fourth point, S , and the middle points of the segments on \mathcal{L} correspond anharmonically to the points where these conics cut the line $Pm'''n'''o'''e'$. Therefore the two conics $PQRSe'$ and $abcde$ cut \mathcal{L} in the same two points, that is, in the given imaginary points.

The construction of the point e' is evidently possible with the straight edge alone.

40. PROBLEM. — To construct a point on a conic that shall pass through a given real point and two given couples of imaginary points.

If the points Q and R are imaginary, the methods of the preceding article enable us to determine the points where the conics $PQRmm'$, $PQRnn'$, and $PQRoo'$ cut a line through S , since three given points of each conic are real. Hence the point e' can be constructed.

41. Hence the points conjugate to two given imaginary points may be constructed.

45. To construct the conjugate to any point, O , in the plane PQR . Draw PO' polar to O with respect to the angle mPn , and draw the straight line aO cutting Pn in h . Construct on Pn the point h' conjugate to h (Art. 44). The point O' , where the straight line PO' cuts the conic $PQRh'a$ is readily constructed, and is the conjugate to O .

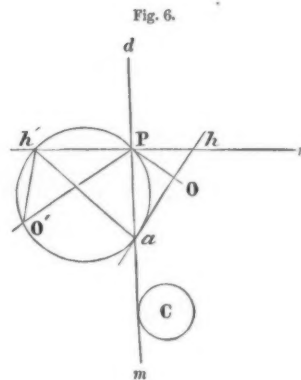
For the conjugate to O is the intersection of the conjugates of the lines PO and aO , that is, the intersection of the line PO' and the conic through P, Q, R , and the conjugates of a and h . But a is conjugate to itself, and h' is conjugate to h .

46. If Q and R are the imaginary circular points at infinity, this method of transforming curves becomes essentially that of reciprocal radii vectores.

For in accordance with the constructions of Art. 42, the point m (Fig. 6) is at infinity on the line Pa , and ad is bisected in P . The conic C is any circle, and the polar of m with respect to C is a diameter perpendicular to ad . Hence Pn , drawn to meet this polar at infinity, is perpendicular to ad . Any circle through a and d will cut Pn in two points, h and h' , conjugate to each other (8), and

$$Pa^2 = -Ph \cdot Ph'.$$

Again, the polar to any point, O , with respect to the right angle nPm , makes the angle $O'Pa = OPa$ (*Géom. Sup.*, Art. 80), and if Oh be drawn and h' be conjugate to h , the point O' , where the circle at aPh' cuts PO' , is the conjugate of O (Art. 45). Now the angle hah' is a right angle, being in a semicircle, and hence $Pha = Pa'h' = h'O'P$. But the angle $h'PO' = hPO$, and therefore the triangles PQh and $P'O'h'$ are similar, and $PO \cdot PO' = Ph \cdot Ph' = Pa^2$.



If then the figure containing O' be revolved 180° about ad as an axis, O and O' will be on the same straight line through P , and $PO \cdot PO' = a$ constant. But these are the relations of two conjugate points in the transformation by reciprocal radii vectores.

47. If we have given two imaginary points, and Q and R are also imaginary, we can construct a line and a conic through the two conjugate imaginary points, thus:—

Construct two points of the conic through P, Q, R , and the two imaginary points (Art. 40). Find the conjugates to these two points (Art. 45), and the line through them is the required line. The conic through P, Q, R , and two points conjugate to any two points of the line that passes through the given points, is the required conic.

48. All these constructions of conjugate points, where either the base points or the given points, or both, are imaginary, can be performed by the straight edge alone without the use of compasses.

49. This method of construction not only applies to curves that shall pass through given points, but in many cases conditions of tangency may also be imposed; for when two curves are tangent to each other, the curves conjugate to them touch each other (Art. 12). Also, if one condition is that there shall be a cusp at a given point, this may sometimes be introduced. For if a curve touch an axis, the conjugate curve may easily be shown to have a cusp at the opposite base point.

50. To avoid repetition, the following notation will be used.

A curve through *given* points will be indicated by letters representing those points. Exponents indicate corresponding multiple points. Thus the cubic, having a double point at the given point a , and passing through the six given points b, c, d, e, f , and g , will be called the cubic $a^2 b c d e f g$.

When dots follow the letters, they denote the number of condi-

tions which the curve can be made to satisfy, besides passing through the given points indicated by the letters. Thus the conic $abc..$ indicates the conic that passes through the given points a , b , and c , and satisfies two other conditions.

51. A few examples of the construction of tangent curves will be given. They can be multiplied indefinitely.

The theory of conics affords means of drawing a line through a given point to touch a conic that passes through five given points; hence may be described :—

- a line a . to touch the cubic $a^2 b^2 c d e f g$;
the biquadratic $a^2 b^2 c^2 d e f g h$;
- a conic $abcd$. to touch the line ef ;
the conic $abefg$;
the cubic $a^2 b c e f g h$;
the biquadratic $a^2 b^2 c^2 e f g h k$.

For the construction may be made to depend on that of a line through a given point, tangent to a conic through five given points, by taking a , b , and c as base points in each case. There are two solutions, for there are two tangents from a point to a conic.

52. A line can be drawn to touch two conics when a common chord of the conics is given. For this by projection may depend on the problem of drawing a line tangent to two given circles, which is an easily solved problem.

Hence may be described a conic $abc..$ to touch,—

- the line de , and the line fg ;
the conic $abfgh$;
the cubic $a^2 b c d f g h$;
the biquadratic $a^2 b^2 c^2 d e f g h$;
- the conic $abdef$ and the conic $abghk$ (or $acdgh$);
the cubic $a^2 b c d g h k$ (or $a b c^2 d e g h$);
the biquadratic $a^2 b^2 c^2 d e f g h$;

the cubic $a^2 b c d e f g$ and the cubic $a^2 b c d h k l$ (or $a b^2 c d e h k$);
the biquadratic $a^2 b^3 c^2 d e h k l$;
the biquadratics $a^2 b^2 c^2 d e f g h$ and $a^2 b^2 c^2 d e k l m$.

53. The construction by points of a conic that shall touch three given straight lines and pass through two given points is possible. Hence can be constructed the biquadratic $a^2 b^2 c^2 d e \dots$ to touch the three conics $a b c f g$, $a b c h k$, and $a b c l m$. One or more of these conics may be replaced by straight lines through double points of the biquadratic, or the curve may be required to have cusps at one or more of the double points. Again, if b , c , and d are in a straight line, the biquadratic is replaced by the cubic $a^2 b c e \dots$. If in addition the points a , c , and e are in a straight line, the biquadratic becomes the conic $a b \dots$. There are four solutions.

54. The most general problem of constructing a conic to pass through given points, touch given straight lines, or given conics, is to construct a conic to touch five given conics. For a point may be considered as an indefinitely small conic, or a conic may be considered as a straight line or as two straight lines. The points may be imaginary, as when the conic is required to be a circle. This general problem is doubtless impossible with the rule and compasses.

Most of the special cases hitherto solved are included in the following problem: *To construct a conic $a b \dots$ to touch the three conics $a b c d e$, $a b f g h$, and $a b k l m$, together with the problem reciprocally polar to it, viz. Given three conics, each of which touches the same two straight lines, to describe a conic tangent to the two lines and three conics.* Solutions of these problems are given by various writers. We mention only PONCELET (*Propriétés Projectives des Figures*, Art. 406) and CHARLES (GERGONNE'S *Annales*, XVIII. 293). There are eight solutions to each problem.

55. Upon the solution of the first of these constructions depends the construction of the biquadratic $a^2 b^2 c^2 d e \dots$ to touch three biquadratics, $a^2 b^2 c^2 d e f g h$, $a^2 b^2 c^2 d e k l m$, and $a^2 b^2 c^2 d e n o p$.

By supposing some of these points on straight lines the problem may be changed in many ways. Thus, if b, c , and d are on one straight line, the construction is that of a cubic $a^2 b c e \dots$ to touch three cubics.

If a, b , and f are on one straight line, b, c , and g on another, and a, c , and h on a third, then the biquadratic $a^2 b^2 c^2 d e f g h$ becomes the line $d e$.

And so in various ways the curves may become cubics, conics, or right lines.

56. The general solution corresponding to that of the second of the problems in Art. 54 is quite complicated. The following special case may, however, deserve mention.

A conic can be drawn to touch five given lines; hence can be described a biquadratic, $a^2 b^2 c^2 \dots$, to touch five conics, $a b c d e$, $a b c f g$, $a b c h k$, $a b c l m$, and $a b c n o$. For any or all of these conics can be substituted straight lines, each passing through a double point, though not more than two lines can pass through the same double point. Instead of one, two, or three of the lines, or conics, there may be required as many cusps at the double points. There is but one solution, and the construction can be performed by the straight edge alone.

57. PONCELET shows (*Prop. Proj.*, Art. 444) that a conic may be constructed that shall have a double contact with a given conic, and that shall touch given lines, or pass through given points, the lines and points to be three in number. The number of solutions in each case is four. These constructions, together with those of the two special cases in the 54th Art., may be made to include all the solutions that I now recollect to have met with of the general problem, *To draw a conic tangent to five given conics.*

58. Since the conic $d e f \dots$ can be drawn (Art. 57) to touch doubly a given conic, it is possible to construct the biquadratic $a^2 b^2 c^2 d e f \dots$ to touch doubly the biquadratic $a^2 b^2 c^2 g h k l m$.

59. If d, e , and f are on the lines ab, ac , and bc severally, then the biquadratic $a^2 b^2 c^2 def..$ is replaced by a straight line; for the biquadratic becomes four straight lines, of which three, ab, ac , and bc , meet $a^2 b^2 c^2 g h k l m$ only in a, b , and c , so that the points of contact must be on the fourth line. Hence can be constructed, with the rule and compasses, *the four double tangents of the biquadratic $a^2 b^2 c^2 g h k l m$.*

60. Again, if g, h , and k are taken on the lines ab, ac , and bc , the biquadratic $a^2 b^2 c^2 g h k l m$ is replaced by the line lm ; hence can be constructed four biquadratics $a^2 b^2 c^2 def..$, of which a given line, lm , shall be a double tangent. In like manner can other special cases of the problem of the 58th Art. be obtained.

61. Since a conic, $de...$, can be drawn to touch a line, fg , and to touch doubly the conic $h k l m n$, it is possible to construct the biquadratic $a^2 b^2 c^2 de...$ to touch the conic $ab c f g$, and to touch doubly the biquadratic $a^2 b^2 c^2 h k l m n$. A score of special cases may be obtained by supposing these curves to break up into lines. If f is on ab , the conic $ab c f g$ is replaced by the line ag . If g is also on ab , the conic becomes the straight line $b c f g$, together with any line through a . The condition of touching such a conic is evidently equivalent to a cusp at c .

62. Because a conic, $d....$, can be drawn to touch doubly a given conic, $e f g h k$, and singly two straight lines, lm and no , therefore can be drawn a biquadratic, $a^2 b^2 c^2 d....$, to touch doubly the biquadratic $a^2 b^2 c^2 e f g h k$, and to touch singly the conics $ab c l m$ and $ab c n o$. In this case the biquadratic $a^2 b^2 c^2 d....$ may be required to be the cubic $a^2 b c....$, the given biquadratic may be a cubic, or a conic, or a right line, or the two conics may be replaced by lines through double points, or by cusps at those points.

63. Again, a conic can be described to touch three given straight lines, and touch doubly a conic; hence can be constructed a biquadratic, $a^2 b^2 c^2....$, to touch three conics, $ab c d e$, $ab c f g$, and $ab c h k$, and

to touch doubly the biquadratic $a^2 b^2 c^2 l m n o p$. A special case is the construction of the biquadratic $a^2 b^2 c^2 \dots$ to touch three given straight lines, and have a fourth given straight line as a double tangent.

64. A faisceau of cubics, all of which pass through eight points (and therefore through a ninth), cuts a right line through one of these points in a series of couples of points in involution (CHASLES, in *Comptes Rendus*, XLI., Dec. 24). The double points of this involution are easily constructed, and are the points of tangency of two cubics passing through the eight points and touching the given line; hence can be constructed a cubic, $abcdefgh.$, to touch a right line hk . Therefore can be constructed a cubic, $abcdefgh.$, to touch a conic, $abchh$; also a biquadratic, $a^2 b^2 cdefghl.$, to touch the conic $abchh$, or the line ak , or the line hl , or the curve may be required to have a cusp at a .

65. In like manner the construction of the biquadratic through fourteen single points enables us to construct the biquadratic through thirteen points, and tangent to a right line through two of the points. This by transformation, as in previous articles, enables us to construct various curves; but they are all of a higher degree than the fourth.

66. It is hardly necessary to add, that in all these cases the points may be imaginary by couples, or, instead of passing through two given points, the curve may be required to touch a line in a given point.

Tangencies of a higher order, with given curves at given points, may in most instances also replace a corresponding number of points.

AWARD OF THE PRIZES FOR SOLUTIONS OF PROBLEMS
IN No. V., Vol. III.

THE first prize is awarded to J. COMFORT, Princeton College, N. J.

THE second prize is awarded to WM. H. EVANS, Marietta College, O.

THE third prize is awarded to A. STANLEY, London, England.

THE fourth prize is awarded to M. C. COREY, Exeter, N. H.

PRIZE SOLUTION OF PROBLEM I.

BY ALL THE COMPETITORS.

Find a point without two concentric circles, from which if two tangents be drawn to the circles the one shall be the double of the other.

Let R and r be the radii of the two circles, x the distance of the required point from the common centre; then $x + R$ and $x + r$ will be the secants, and $x - R$ and $x - r$ the external segments; and since the tangent is a mean proportional between the secant and its external segment, we have $2\sqrt{(x^2 - R^2)} = \sqrt{(x^2 - r^2)}$, whence

$$x = \sqrt{\left(\frac{4R^2 - r^2}{3}\right)}.$$

PRIZE SOLUTION OF PROBLEM II.

By A. STANLEY, London, England.

Any whole number and its fifth power, when divided by 30, leave the same remainder; also, the number and its seventh power, when divided by 42, leave the same remainder.

Whatever whole number greater than 2 (when $n = 2$, the truth of the theorem is evident) n may represent, the product

$$(1) \quad (n - 2)(n - 1)n(n + 1)(n + 2)$$

is exactly divisible by the product $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$, since the quotient represents the number of combinations of $n + 2$ things in groups of 5. Similarly, $(n - 1)n(n + 1)$ is divisible by $1 \cdot 2 \cdot 3$, so that the

product (1), as well as the product $5(n-1)n(n+1)$, and hence also their sum, must be divisible by $1 \cdot 2 \cdot 3 \cdot 5 = 30$; but this sum is clearly equal to $(n-1)n(n+1)(n^2+1) = n^5 - n$; whence we conclude that n^5 and n give the same remainder when divided by 3.

In a similar manner, it may be shown that n^7 and n give the same remainder when divided by $1 \cdot 2 \cdot 3 \cdot 7 = 42$.

PRIZE SOLUTION OF PROBLEM III.

There are n straight lines making, with another fixed straight line, the angles $\alpha, \beta, \gamma \dots$; a point, P , is taken, such that the sum of the squares of the perpendiculars from it on these n lines is constant. Find the conditions that the locus of P may be a circle.

The equations of these n straight lines are

$$x_1 \cos \alpha + y_1 \sin \alpha - \delta_1 = 0, \quad x_2 \cos \beta + y_2 \sin \beta - \delta_2 = 0, \text{ \&c.,}$$

and the n perpendiculars, dropped from $P(x, y)$ on these lines are (SALMON'S Conic Sections, p. 21)

$$\pm(x \cos \alpha + y \sin \alpha - \delta_1), \quad \pm(x \cos \beta + y \sin \beta - \delta_2), \text{ \&c.}$$

The sum of the squares of these perpendiculars must equal a constant, and in order that the locus of the equation thus formed shall be a circle, the coefficients of x^2 and y^2 must be equal, and the coefficient of xy must equal zero. These conditions readily lead to

$$(1) \quad \cos 2\alpha + \cos 2\beta + \cos 2\gamma + \dots = 0,$$

$$(2) \quad \sin 2\alpha + \sin 2\beta + \sin 2\gamma + \dots = 0.$$

This is essentially the solution given by all the competitors.

PRIZE SOLUTION OF PROBLEM IV.

By J. COMFORT, Princeton College, N. J.

If A and B represent the semi-axes of an ellipse, the altitudes of the minimum circumscribing isosceles triangles, having their vertices in the axes produced, are as $A : B$.

From the equation of the tangent $A^2 y y' + B^2 x x' = A^2 B^2$, we get $y = \frac{B(A+x')}{\sqrt{(A^2-x'^2)}}$ when $x = -A$, and $x = \frac{A(A+x')}{x'}$ when $y = 0$. But the area of the triangle having its vertex in the transverse axis produced is $xy = \frac{AB(A+x')^2}{x'\sqrt{(A^2-x'^2)}} = u$. Let

$$u = 2 \log (A + x') - \log x - \frac{1}{2} \log (A^2 - x'^2),$$

then

$$\frac{du}{dx'} = \frac{2}{A+x'} - \frac{1}{x'} + \frac{x'}{A^2-x'^2} = \frac{2A-x'}{A^2-x'^2} - \frac{1}{x'} = 0.$$

$$\therefore x' = \frac{A}{2}, \text{ and } x = \frac{A(A+x')}{x'} = 3A.$$

Then $x' = \frac{A}{2}$, $\frac{d^2u}{dx'^2} = \frac{76}{9}A^2$, $\therefore u$ is a minimum.

In like manner, the altitude of the other triangle may be shown to be $3B$. Hence their altitudes are as $A : B$.

PRIZE SOLUTION OF PROBLEM V.

By WILLIAM H. EVANS, Marietta College, Ohio.

Prove that, of all circular sectors having the same perimeter, the one of greatest area is that in which the circular arc is double the radius.

Let r denote the radius of a sector whose arc is $2r$. The perimeter will be $4r$, and area r^2 . If the radius be diminished by any quantity, h , the perimeter remaining the same, the arc becomes $2r + 2h$, and the area $= r^2 - h^2$. But if the radius be increased by any quantity, h , then, the perimeter remaining the same, the arc becomes $2r - 2h$, and the area $= r^2 - h^2$, and in both cases the area r^2 is diminished. Hence the proposition is proved.

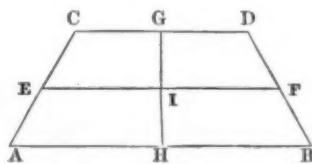
DEMONSTRATION OF THE LAW OF EQUILIBRIUM IN THE LEVER.

By DR. THEODORE STRONG, Professor of Mathematics in Rutgers College, New Brunswick, N. J.

THE principle on which the demonstration is based is, that for any proposed weight or power applied at any point of any straight lever of the first kind, we may always substitute two equal weights or powers, each equal to half the weight or power acting on opposite sides of the point, at equal distances from it, and in a way similar to that of the proposed weight or power.

That the principle is true, may be shown in the following manner.

Let $ABCD$ represent a quadrilateral, having its opposite sides, AB and CD , parallel to each other, and having equal weights or powers acting in like manner at each of its angles. Then, if the straight line EF bisects the opposite sides, AC and BD , the quadrilateral will be in equilibrium on EF , regarded as an axis. For



it is clear that AC may be regarded as a straight lever of the first kind, acted on by the equal forces at A and C , which tend to turn it in opposite directions around its middle point, E , regarded as a fulcrum; consequently, since there is no reason why either of the forces should prevail over the other, they must be in equilibrium on the point. Since it may be shown, by a like reasoning, that the forces at B and D will be in equilibrium on the point F , it clearly follows that the figure is in equilibrium on the straight line EF .

If the straight line GH bisects the parallel sides AB and CD , a like reasoning will show the figure to be in equilibrium on the right line GH . Hence, because the forces balance each other on the right lines EF and GH , it is evident that they must balance each other

on the point I , the common section of these lines. Consequently, since I is the middle point of GH , it may be regarded as a lever of the first kind, in equilibrio on its middle point, I ; and hence the action of the forces A and B on the point H must equal the action of the forces C and D on the point G .

Supposing the distances AH and BH , as well as the forces at A and B , to be invariable, while the distances $CG = GD$ are variable; then it is manifest that the action of the forces at C and D must be invariable. Hence, if $CG = GD = 0$, it follows that the force at G is always equal to the sum of the forces at C and D , or to twice either of them; and, reciprocally, the force at C or D is half the force at G . See p. 5, Vol. I., of the *Mécanique Analytique* of LAGRANGE.

A P' W C P'' P B

Supposing AB to be a straight lever of arbitrary length, regarded as without weight; W and P the points of the lever where the weight and power are similarly applied to it, having the fulcrum on the point on which the lever is supported between them, as in the lever of the first kind. Then, according to the law of equilibrium in the lever, if W has the same ratio to P that PC has to WC , the weight (W) and power (P) will balance each other on the fulcrum (C).

For suppose n to stand for any positive whole number, and that n equals the ratio of W to P or that of PC to WC ; then we shall have $W = nP$, and $PC = WC \times n$. Take $WC \times (n - 1)$ and set it off on each side of W to the points P' and P'' on the lever.

According to the preceding principle, we put $(n - 2)P$ at W , P at P' , and P'' , for $W = nP$, at W ; consequently, since $P'C = PC$, P at P is destroyed by the contrary action of P at P' , so that, without affecting the equilibrium, we have changed $W = nP$ and P acting at the distances WC and $PC = nWC$ from the fulcrum,

into $W - 2P = (n - 2)P$ and P , acting at the distances WC and $P''C = (n - 2)WC$ from the fulcrum, noticing that these changes consist merely in changing n into $n - 2$. It is easy to perceive that (if necessary) we may make like changes in the preceding results, and so on, until (when n is an even number) the force or weight at W vanishes, and the remaining force P is destroyed by the reaction of the fulcrum; or (when n is an odd number) until the force P at W is destroyed by the equal force P at the distance WC on the opposite side of the fulcrum, so that when n is a positive integer (as supposed) the proposed weight and power will be in equilibrio on the fulcrum.

Thus, if $n = 2$, we have $W - 2P = (n - 2)P = 0$, and $P''C = (n - 2)WC = 0$; and the weight at W is reduced to naught, and the power P at P'' is destroyed by the reaction of the fulcrum.

Also, if $n = 3$, the weight at W is reduced to $(n - 2)P = P$, which is manifestly destroyed by the contrary action of the power P at the distance $P''C = (n - 2)WC = WC$ from the fulcrum.

If m stands for any positive number, it clearly follows, from what has been done, that m times the power P acting at the extremity P of PC must balance m times the weight W acting at the distance WC from the fulcrum.

Hence, we must have $mP \times PC = mW \times WC$; and if we have in like manner the power $m'P'$ acting on the lever $P'C'$, so as to balance the weight mW by the lever WC , then, as before, we shall have $m'P' \times P'C' = mW \times WC$.

Hence, by eliminating $mW \times WC$ from these equations, we have $mP \times PC = m'P' \times P'C'$.

This equation shows that the arms of the straight lever of the first kind when in equilibrio are *inversely as the forces that are applied to them*.

Mathematical Monthly Notices.

The Mathematical Works of ISAAC BARROW, D.D., Master of Trinity College, Cambridge.
 Edited for Trinity College by W. WHEWELL, D.D., Master of the College. Cambridge:
 Printed at the University Press. 1860. (From the Editor.)

The very best possible notice of the work before us is contained in the learned Editor's Preface, which we present to our readers entire.

"It was thought right, by BARROW's College, that a new edition of his Mathematical Works should accompany the edition of his Theological Works lately published by the University under the editorial care of Mr. NAPIER; and I have willingly undertaken to superintend the printing of the edition thus agreed upon. I have already, in the Preface to Vol. IX. of Mr. NAPIER's edition, given an account of BARROW's mathematical writings; but for the sake of convenience I will here resume the subject.

"BARROW's first mathematical publication appears to have been his edition of EUCLID. I have supposed, in my former Notice, that this was published in 1654, before he set out upon his travels. But the first edition is dated 1655. It is probable that he left the manuscript to be printed after his departure; for in the Preface he says that he has tried to reduce the book into a small space: '*Id quod assecutus videor, si absentem Typographi cura non frustratur.*' This edition contains all the fifteen Books of EUCLID's Elements.* I shall not insert the bulk of this publication in the present edition; but at the end of this Prefatory Notice, I will insert BARROW's Dedication and Preface to this book. The Dedication is addressed to three young men, who, from the terms in which they are spoken of, must have been, I conceive, BARROW's pupils. They were EDWARD CECIL, son of the EARL OF SALISBURY, JOHN KNATCHBUL, and FRANCIS WILLOUGHBY.† The latter was afterwards the celebrated reformer of Ichthyology, the friend of RAY, the botanist. He addresses these young men with expressions of great affection and esteem; and says that no one can know their good qualities better than himself, in virtue of the sweet habitual intercourse which he has had with them. In his Preface he speaks of the two main objects which he had in editing EUCLID, to reduce the whole of the Elements into a portable volume, and to gratify those readers who prefer symbolical to verbal reasoning. He would have been satisfied, he says, with TACQUET's edition, if TACQUET had not confined it to eight books (the I., II., III., IV., V., VI., XI., and XII.), omitting the remaining seven. The symbols which he has used are mostly those of WILLIAM OUGHTREED, 'to which most of us are accustomed.' These are the usual algebraical symbols, of which the introduction was then recent.

"But the principal contents of the present volume are the Lectures which BARROW delivered as Lucasian Professor of Mathematics; an office which he held from 1664 to 1670. And these consist of three series: the *Lectioes Mathematicæ*, the *Lectioes Opticæ*, and the *Lectioes Geometricæ*; the first being on the general Principles of Mathematics, the second containing propositions of Optics proved geometrically, and the third treating of properties of Curve Lines. I will make a few remarks on some of these Lectures.

"The first or Inaugural Lecture‡ contains an account of HENRY LUCAS, the founder of the Professorship, given in BARROW's usual rhetorical manner. He begins by referring to the tranquillity which had been restored to the nation by the cessation of the civil wars. 'The turmoil of public business being reduced to rest, and the restoration of tranquillity having given you some heart to tell and to hear news, listen to me, Academics, while I haste to tell you something strange and almost a prodigy. There has shone forth of late—What? you will say. Some dire comet, the presage of calamity, such as are seen in numbers every day (in spite of the heavens themselves) by the distorted vision of fanatics? No: but a new and benignant star, shining with a ray both true and propitious, such as has not for many years risen above the academical horizon. And it is that I may measure its magnitude, explain its motions, and interpret its presages, that I now come forwards, no vain astrologer I.' It is

* EUCLIDIS *Elementorum Libri XV.* breviter demonstrati, Operâ Is. BARROW, Cantabrigiensis, Coll. Trin. Soc. MDCLV. There are many later editions in Latin and in English.

† The entries of these names in the College Admission Book are,—

(P. 23.) EDWARDUS CECIL, filius H. D. Comitiss Sar. admissus commensalis Nov. 30, 1652.

(P. 2.) JOHANNES KNATCHBUL, Cantianus, admissus commensalis May 6, 1652.

(P. 23.) FRANCISCUS WILLOUGHBY, Warwicensis, admissus commensalis Sept. 9, 1652.

All the three being admitted as pupils of Mr. DUPORT.

‡ Though this Lecture is printed in the ninth volume of Mr. NAPIER's edition, I have reprinted it here as the first of the Lucasian Lectures; and have also ventured to repeat in this Preface some of the remarks which I made in the Preface to that volume.

by this image, and others in the same strain, that he describes LUCAS, and his good deeds towards the University. The Lucasian Professorship which he founded he endowed with the rents of an estate in Bedfordshire, amounting at present to £155. BARROW mentions the principal circumstances of LUCAS's life: stating that he had studied at St. John's College, and was then taken into the family of the EARL OF HOLLAND (who was Chancellor of the University from 1625 to 1648), as his secretary. Here, by attention to business and economy, he accumulated a considerable fortune; and continuing unmarried, resolved to make posterity his heir. He represented the University in Parliament from 1640; and as his end approached, considered and consulted with his friends how he could best promote its interests. The result was, that he determined to give encouragement to mathematical studies, which, though admired always, and especially in recent times, had hitherto enjoyed no patronage in the University. BARROW goes on to extend his praises to the two trustees of LUCAS's will: ROBERT RAWORTH, a lawyer, and THOMAS BUCK, a resident in the University, whom he refers to as well known to his hearers; 'the same whose stately presence and dignified countenance is every day before your eyes'; and to him he ascribes the suggestion of LUCAS's benefaction. These two, in conjunction with the Heads of Colleges, and especially the Vice-Chancellors of 1663 and 1664, carefully and diligently took the proper steps for carrying the bequest into effect.

"He then proceeds to speak of his own tastes, and of his purposes with regard to lectures, as I have stated in the notice of him already referred to; and ends his praise of mathematics in language which, as I have said, may remind us of the expressions of FRANCIS BACON.

"The Lectures which in the present volume next follow this Prefatory Lecture are those which were the earliest delivered, namely, in 1664, 5, and 6, but the latest published, namely, not till 1685. They treat, as I have already said, of the general principles and arrangements of mathematics, with a notice of some of the leading controversies and criticisms which had appeared on the subject, extending through twenty-three lectures, and ending with a vindication of EUCLID's Doctrine of Proportion. I have annexed to these, at the foot of the page, a brief summary of their contents, which may enable the English reader to trace the general course of the argument. They display great metaphysical subtlety, logical precision, and a large acquaintance with the literature of mathematics, both ancient and modern. As I have stated in the former account of BARROW, an English translation of these Lectures was published in 1734; but so badly executed that it cannot be of use to any one. The Lectures themselves are of interest to those who love to dwell on the metaphysical grounds of mathematical truths, but do not belong to that progressive line of mathematical speculation to which BARROW had referred in his Prefatory Lecture when he spoke of GALILEO, GASSENDI, GILBERT, MERSENNE, CARTESIUS, and others. BARROW's contributions to this kind of mathematics appear in his *Lectiones Opticæ* and *Lectiones Geometricæ*, as we shall see.

"But next after the *Lectiones Mathematicæ*, I have printed four Lectures, in which he proposes to himself, as he says, to expound the method by which ARCHIMEDES invented his beautiful theorems (those, namely, concerning Cones and Spheres, their Solid Contents and Surfaces). This he says he will do by reducing the steps to problems, such as ARCHIMEDES proposed to himself, and from the solution of which he deduced both his theorems and the mode of demonstrating them: whence, he says, it will appear what was the analysis which he used, and how like our modern analysis it was. It is a thought which has often suggested itself to the readers of Ancient Greek mathematicians, and especially of ARCHIMEDES, that those writers must have been led to their geometrical theorems and proofs by some methodical analytical process. Their constructions and propositions are so complex, recondite, and abstruse, that it seemed impossible that any one should be led to them by mere direct exercise of ingenuity or felicity of conjecture. We know that some modern mathematicians, particularly NEWTON, have followed this practice of discovering a proposition by analysis, and then proving it by a geometrical synthesis. BARROW has, in the four Lectures here given, proved, partly by the use of algebraical analysis, some of the most difficult of the Propositions of ARCHIMEDES concerning Spheres and Cones: for instance, that which forms the last proposition in these Lectures, that of all segments of spheres, of equal extent of surface, the hemisphere has the largest content: a proposition which might be found difficult to demonstrate by a good mathematician of the present day. And as a previous step, he has to solve this problem (Lect. XXVII. Prob. IV.): To cut a given sphere into two segments which have a given ratio. He obtains, by algebraical processes, a certain proportion which gives the point of section of the axis; and which is, he says, the very proportion to which ARCHIMEDES reduces the problem; and this, he says, shows what kind of analysis that author must have employed. For that he got at it by employing the various compositions, divisions, permutations, and inversions of proportion, in the same order in which he presents them in his text, is beyond belief. If he had done this, his lighting upon the right solution would have been rather a matter of chance than of reason or skill; and that this should have happened so constantly, is inconceivable and impossible.

"BARROW's love of the ancient Greek geometers, and his desire to abridge and simplify their demonstrations by introducing into them analysis, led him at a later period to publish an edition of the works of ARCHIMEDES, APOLLONIUS, and THEODOSIUS.*

* ARCHIMEDIS *Opera*; APOLLONII Pergæi *Conicorum Libri quatuor*; THEODOSII *Sphærica Methodo novo illustrata et succinctè demonstrata*. Per IS. BARROW, exprofessorem Lucasianum Cantab. et Societatis Regiæ Soc. 1675.

"To include this work among BARROW's Mathematical Works would have made this publication too bulky: I have printed, at the end of these prefatory remarks, the brief preface which he prefixes to it. His edition contains all the extant works of ARCHIMEDES; namely, the Two Books on the Sphere and Cylinder, agreeing in substance with the four Lectures here given; The Treatises on the Measurement of the Circle; On Spirals; On Conoids and Spheroids; On the Centres of Gravity in Plane Figures; On the Quadrature of the Parabola; On Floating Bodies; On Numbering the Sand. Also his Lemmas, which include, among other propositions concerning tangencies of circles, the proposition concerning the figure which he calls the *Arbelon*, a figure bounded by three semicircles, and named from its resemblance to a leather-cutter's knife.

"Of APOLLONIUS, this edition by BARROW contains only four Books, the only ones which have come down to us in Greek. Books V., VI., and VII. were afterwards found to exist in an Arabic translation; were brought from the East by GOLIVS, translated by ABRAHAM ECHELLESIVS, and published by BORELLI in 1661. The Eighth Book has never been found, but has been restored conjecturally by HALLEY and by VIETA.

"The Three Books of the Spherics of THEODOSIVS contain various propositions concerning the circles of the sphere, small circles as well as great. The propositions here contained are the basis of Spherical Trigonometry; but of course the ancient Greek geometer does not attempt the solution of spherical triangles.

"The Opticæ Lectiones, as I have mentioned in my former account of BARROW, are noticed by historians of mathematics as an important work. In his Optical Speculations, says MONTUCLA, BARROW quitted the beaten track, and discussed questions hitherto imperfectly treated; as the theory of the foci of spherical surfaces and lenses, the apparent places of images, and the like. He also explains the rainbow, simplifying CARTESIUS's calculations; and, seeing that the Cartesian explanation of the colors is not satisfactory, proposes one of his own (Lect. XII. Art. XVI.). It remained for NEWTON to give the true explanation. In the Epistle to the Reader, he states that ISAAC NEWTON had revised and corrected the copy, and added matter of his own; and that COLLINS (who, he says, may be called the *Mersenne* of England, for his merits in promoting mathematical science both by his own labors and those of others) had superintended the edition with great attention. The mathematical reader will recollect that MERSENNE was a correspondent of most of the scientific men of his time, and a centre of correspondence among them.

"The Geometricæ Lectiones are full of curious methods of determining the areas and tangents of curves, many of which are very close anticipations of NEWTON's methods. The most noted of these is the method of drawing tangents to curves, given in Lect. X., Art. XIV. This method is justly held to be an anticipation of the Differential Calculus, and to approach very near to it. It will be best explained by taking BARROW's first example, which is this:—

"In Fig. 116, ABH is a right angle, K any point in BH ; A being a fixed point, AK is joined; and in it, AM is taken equal to BK : it is required to draw a tangent to the curve AM .

"If, as in modern notation, we draw an ordinate, MP , perpendicular to the line of abscissas, AB , and call AP and PM , x and y respectively, AB being $=r$, it is evident that $BK = \frac{r y}{x}$, and $AM = \sqrt{x^2 + y^2}$; whence the equation of the curve is $x^2 + y^2 = \frac{r^2 y^2}{x^2}$; or, $x^4 + x^2 y^2 = r^2 y^2$. And if we differentiate this we obtain, for the subtangent PT ,

$$\frac{y dx}{dy} = \frac{r^2 y^2 - x^2 y^2}{2x^2 + x y^2}.$$

"BARROW's mode of proceeding is this (he uses p and m for x and y , which I shall alter so as to fall in with modern notation):

"Take an ordinate and abscissa near to x and y , and let these be $x-e$ and $y-a$, e and a being small. Therefore

$$(x-e)^2 + (y-a)^2 = A Q^2 + Q N^2 = A N^2 = B L^2 = x^2 - 2xe + e^2 + y^2 - 2ay + a^2.$$

"But

$$A Q : Q N :: A B : B L;$$

that is,

$$x-e : y-a :: r : B L,$$

whence

$$B L^2 = \frac{r^2 y^2 + r^2 a^2 - 2r^2 y a}{x^2 + e^2 - 2xe};$$

and, rejecting superfluous terms (1), in these values of $B L^2$, they are

$$= \frac{r^2 y^2 - 2r^2 y a}{x^2 - 2xe} \quad \text{and} \quad x^2 - 2xe + y^2 - 2ay.$$

"And, equating these, and multiplying up,

$$r^2 y^2 - 2r^2 y a = x^4 - 2x^2 e + x^2 y^2 - 2x^2 y a - 2x^2 e^2 + 4x^2 e^2 - 2xy^2 e + 4xy a e.$$

"That is, rejecting the terms which our rule rejects (2),

$$-2r^2ya = -4x^2e - 2x^2ya - 2xy^2e,$$

or

$$r^2ya - x^2ya = 2x^2e + xy^2e;$$

or, putting the ordinate y and the subtangent t for a and e , (since they are in the same proportion as those lines,)

$$r^2y^2 - x^2y^2 = 2x^2t + xy^2t,$$

whence

$$\frac{r^2y^2 - x^2y^2}{2x^2 + xy^2} = t = PT.$$

"The peculiarity of the method consists in the steps which I have marked (1) and (2); that is, the rejection of superfluous terms. And the rule given by BARROW is this:—

"After constituting the equation to the curve, put $x = a$ and $y = e$ for the ordinates x and y : expand, and reject all the terms in which there is no a or e (for they destroy each other by the nature of the curve); reject all the terms in which a or e are above the first power, or are multiplied together (for they are of no value compared with the rest, as being infinitely small): then put y for a , and t , the subtangent, for e ; and PT is found.

"It is plain that the terms thus retained are the terms involving the first powers of a and e , when, in the equation to the curve, $x = a$ and $y = e$ are put for x and y , and the equation is expanded. But the ratio of the coefficients of these terms is the ratio of dx to dy in the differential calculus: and hence the substantial identity of the two methods is evident. What remained was, to devise a notation, and to assign general rules of obtaining those coefficients.

"BARROW applies this method to the following curves (I use the modern notation for the co-ordinates):—

"Ex. II. The curve $x^2 + y^2 = r^2$.

"Ex. III. The curve $x^2 + y^2 = rxy$, which it appears was called *La Galande*.

"Ex. IV. The *Quadratrix*, of which the equation is

$$y = r - x \tan \frac{\pi x}{2r}.$$

"Ex. V. The curve in which the abscissa being equal to an arc of a circle, the ordinate is equal to its trigonometrical tangent:

$$y = r \tan \frac{\pi x}{2r}.$$

"Also we may regard BARROW's mode of finding the areas of curves by comparing them with the sum of the inscribed and circumscribed parallelograms (Lect. XII., Append. II., Fig. 175, 176), as leading the way to NEWTON's method of doing the same, given in the First Section of the Principia.

"I have, for the most part, retained BARROW's notation, with slight alterations where it would have been likely to mislead a modern reader. Thus where we write $A : B$, he writes $A \cdot B$; and where we write $A > B$, he writes $A \sqsubset B$. I have retained Aq for A^2 , $A \text{ cub}$ for A^3 , Aqq for A^4 , and the like.

"It is a matter of labor and difficulty for a reader in these days to follow out the complex constructions and reasonings of a mathematician of BARROW's time; and I do not pretend that I have in all cases gone through them to my satisfaction. I have however, in several cases, endeavored to assist my reader to follow these demonstrations with moderate trouble."

Elements of the Differential and Integral Calculus. Prepared by ALBERT E. CHURCH, LL. D., Professor of Mathematics in the U. S. Military Academy. Revised Edition, containing the Elements of the Calculus of Variations. New York: Published by A. S. Barnes and Burr, 51 and 53 John Street. 1861.

The earlier editions of this work are so well known to teachers and students, that an extended notice would seem unnecessary. The author says: "An experience of more than twenty-five years in teaching large classes in the United States Military Academy has afforded the author of the following pages unusual opportunities to become familiar with the difficulties encountered by most pupils in the study of the Differential and Integral Calculus." A careful examination of this edition must convince the experienced teacher that it realizes all the essential requisites of a good text-book. The demonstrations are clear and rigorous, the solutions are carried out in sufficient detail to enable the student to grasp the principles involved, the

text is written with care and accuracy, and all the parts, including the Calculus of Variations, are treated as fully as would be proper in an elementary work. It is possible that some teachers may find the number of problems too small; but this want is easily supplied by the excellent collections of examples already well known. But, as in the United States Military Academy, an immediate application of the subject to professional details renders a large number of examples unnecessary; and this consideration has probably guided the author in the number selected.

The earlier part of the volume is based on the doctrine of limits; but on page 127 the infinitesimal method is introduced, and retained throughout the volume. This certainly gives rise to the suggestion, whether it would not have been better to have adopted the latter method at the beginning; but this is, without doubt, one of the points which has received the author's particular attention, and should not, therefore, be hastily condemned. We close by expressing the conviction that teachers will find this edition well adapted to the wants of beginners, and in most respects an excellent text-book.

Elementary and Practical Arithmetic; in which have been attempted various Improvements in Arrangement and Nomenclature, and in the Means of thorough Discipline in the Principles and Applications of the Science. New York: Pratt, Oakley, & Co., 21 Murray Street pp. 291.

Arithmetic for High Schools; containing the Elementary and Higher Principles and Applications of the Science. New York: Pratt, Oakley, & Co. pp. 336.

Elementary and Practical Algebra; in which have been attempted Improvements in General Arrangement and Exposition. New York: Pratt, Oakley, & Co. pp. 220.

Algebra for High Schools and Colleges; containing a Systematic Exposition and Application of the Elementary and Higher Principles of the Science. New York: Pratt, Oakley, & Co. pp. 306.

Elements of Geometry and Mensuration. New York: Pratt, Oakley, & Co. pp. 237.

Elements of Trigonometry, Plane and Spherical; with Applications to Heights and Distances, the Areas of Polygons, Surveying, Navigation, and the Solution of Astronomical Problems. New York: Pratt, Oakley, & Co. pp. 292.

This series of text-books has been prepared by JAMES B. DODD, A. M., Professor of Mathematics and Natural Philosophy in Transylvania University. Our want of space prevents a detailed notice of each of the above volumes, and we shall therefore confine ourselves to some general remarks on some of the more striking features of the series.

In the first place, the Arithmetics are precisely the same as far as page 261, and the Algebras as far as page 220. The Elementary Arithmetic, in addition, contains thirty pages on geometrical definitions, and rules for the mensuration of geometrical figures; the High-School Arithmetic, in addition, the Progressions, Compound Interest, Annuities, Permutations, Combinations, Exchange, &c. In addition to the 220 pages comprising the Elementary Algebra, the High School Algebra contains four chapters, viz. General Theory of Equations, Continued Fractions, Logarithms and their Applications, Infinite and Infinitesimal Quantities, and Infinite Series; together with a Table of Six-place Logarithms from 1 to 10,000.

This kind of gradation in a series of books on the same subject has, it seems to us, some decided advantages. We have examined the Arithmetics and Algebras with care, and while there are some points in which we think they could be improved, they are by no means inferior books. Teachers will find that they have not been ruined by any attempts at originality to which the author was unequal, but his aim has clearly been to present the subjects, both with respect to style and arrangement, in the clearest and simplest manner.

While the Geometry as a whole impresses us favorably, we are especially pleased with

Book X., on the Maxima and Minima of Geometrical Figures, as a valuable improvement upon the works on geometry commonly used in our schools. This book will prove an excellent test of the student's appreciation of the true spirit of the subject, and be a fitting close of his studies in elementary geometry.

The Trigonometry contains, in our opinion, one serious defect, in making the values of the trigonometrical functions depend upon the length of the radius, when they do in fact depend solely upon the magnitude of the angle to which they relate. This method has been abandoned in all the best modern text-books, both at home and abroad, not only as unphilosophical, but as being far inferior for the simple purpose of instruction, and especially embarrassing to the student in reading the modern works on the higher analysis.

We are, however, especially pleased with the method and clearness with which all the applications are treated; and, as an example, would call the reader's attention to the treatment of Division of Land, or of Polygons in general.

Finally, we say that teachers, in deciding upon the adoption of a series of text-books upon the elementary mathematics, will do their pupils, as well as the author, great injustice, if they fail to give this series a fair and impartial consideration.

The Quadrature of the Circle. Correspondence between an Eminent Mathematician and JAMES SMITH, Esq., (Member of the Mersey Docks and Harbour Board.) Author of "The Question, Are there any Commensurable Relations between a Circle and other geometrical Figures? answered by a Member of the British Association for the Advancement of Science." "*Strike, but Hear!*" London: Simpkin, Marshall, & Co., Stationers' Hall Court. Edinburgh: Oliver and Boyd. 1861. pp. 200.

This is the third instance which has come to our knowledge, within the past couple of years, of an attempt to "Square the Circle." In the other cases, the authors plead guilty, in less pretentious volumes, to all of JAMES SMITH's impenetrable stupidity; but they failed in the possession of that sublime impudence which makes JAMES SMITH a character not altogether devoid of interest, and mitigates in some degree the regret that he should be embalmed in so fine a specimen of typography.

It seems that the author issued a pamphlet bearing the title quoted above, and that an amiable gentleman of London attempted to point out the fallacy into which he had fallen; but soon found the attempt useless, as the author would not even admit the evidence of his senses, that by actual measurement his ratio of $3\frac{1}{2}$ was not the true one. "Eminent Mathematician," as JAMES SMITH styles his correspondent, declines, in a remarkably forbearing and courteous manner, after finding that he had "caught a Tartar," to continue the correspondence. But his horror may easily be conceived on receiving the letter which hinted at his unsought immortality. We give his reply, with JAMES SMITH's matchless answer.

"EMINENT MATHEMATICIAN TO JAMES SMITH, Esq.

"London, 15th December, 1860.

"SIR, — I have received your letter of the 30th Nov., delivered here on the 11th December.

"With reference to the words in your letter, 'previously to publishing the entire correspondence,' I must be allowed to make a remark.

"My letters to you were written in the sincere conviction that I was writing to one earnestly engaged in the search after truth, and my observations were confined to the pointing out to him how he might convince himself that he was altogether wrong. My letters were not intended for publication, and I protest against their being published, for I do not wish to be gibbeted to the world as having been foolish enough to enter upon, what I feel now to have been, a ridiculous enterprise.

"Therefore, I must desire that my name may not be used.

"I am, Sir, obediently yours,

"E. M.

"JAMES SMITH, ESQ. TO EMINENT MATHEMATICIAN.

"*Barkeley House, Seaforth, Liverpool, 15th Dec., 1860.*

"SIR, — I have the honor to acknowledge the receipt of your favor of the 13th inst.

"You object to my publishing our correspondence on the 'The Quadrature of the Circle,' and give your reasons for it.

"You first say, '*Your letters to me were written in the sincere conviction that you were writing to one earnestly engaged in the search after truth.*'

"May I request you to point out a paragraph throughout our correspondence, from which you can reasonably draw an opposite conclusion?

"You then say, '*Your observations were confined to pointing out to me how I might convince myself that I am altogether wrong.*'

"I am willing to admit that this was your intention, and if in our correspondence you have succeeded in accomplishing what you proposed to yourself, may I ask, Of what have you to be ashamed if the correspondence be published?

"You next say, '*Your letters were not intended for publication.*'

"I can as truly say, that, to a certain point in our correspondence, neither were mine.

"You then protest against your letters being published, and give as your reason, '*That you do not wish to be gibbeted to the world as having been foolish enough to enter upon, what you feel now to have been, a ridiculous enterprise.*'

"This sentence is capable of two interpretations. It may mean, that in an evil moment you were foolish enough to enter into a correspondence with a man of such ignorance and stupidity that no amount of legitimate argument could possibly influence his silly judgment. Or, it may mean, that you undertook a task which you find yourself unable to accomplish, and that in doing so you were foolish enough to enter upon, what you feel now to have been, a ridiculous enterprise.

"If the former be your opinion, I can see no reason why you should have the least objection to the publication of our correspondence, for in that case I should be the party gibbeted, not you.

"If the latter be your opinion, you have certainly not had the candor to distinctly avow it.

"Pray inform me which alternative you wish me to accept as your meaning, and I will then tell you what course I shall adopt, and give you my reasons for it.

"I am, Sir, yours very respectfully,

"JAMES SMITH."

Those wishing to study JAMES SMITH further must buy his book. It teaches one important lesson, which is sufficiently obvious without an explicit statement.

Editorial Items.

Solutions of problems in the January number have been received from MISS H. S. HAZELTINE, Worcester, Mass.; GEORGE W. PIERCE, Freshman Class, Harvard College; ASHER B. EVANS, Madison University, Hamilton, N. Y.; JAMES G. MARSTON, Military Institute, Franklin County, Ky.; DAVID TROWBRIDGE, Perry City, N. Y.; JOHN Q. HOLLISTER, Hamilton College, Clinton, N. Y.; W. K. LEMMON, McKendree College, Lebanon, Ill.; H. J. McMURPHY, Londonderry, N. H.; SAMUEL MANN, Athol, Mass.; WILLIAM TIMPSON, White Plains, N. Y.; MARCUS M. RHOADES, Mount Pleasant College, Huntsville, Mo.; JAMES F. ROBERSON, Grantsburgh, Ind.; J. M. TERRILL, Linneus, Mo.; E. O. GIBSON, South New Berlin, N. Y.; GEORGE B. HICKS, Cleveland, Ohio.

Solutions of problems in the February number have been received from JAMES CLARK, Wayne, Me.; BOON GRIFFIN, McKendree College, Ill.; JOHN A. WINEBRENER, Princeton College, N. J.; DAVID TROWBRIDGE; WILLIAM H. EVANS, Marietta College, Ohio; E. CANFIELD, Chester, N. J.; M. K. BOSWORTH and HIRAM L. GEAR, Marietta College, Ohio; C. H. WILLARD, Antioch, Ill.; A. G. BARKER, Waterville College, Me.; PHILLO HOLCOMB, Cardington, Ohio; J. F. ROBERSON; GEORGE B. HICKS; A. STANLEY, London, England; J. COMFORT, Princeton College, N. J.; M. C. COREY, Exeter Academy, N. H.; ASHER B. EVANS, Nunda, N. Y.; E. O. GIBSON, South New Berlin, N. Y.

June,

Worcester's Quarto Dictionary The Standard

VERY SIGNIFICANT FACTS.

The following recommendations are from some of the most distinguished American and English scholars. They are but a few from many which have been received, testifying to the superiority of Worcester's Quarto Dictionary. These testimonials are of the highest value, for they have all been given during the present year, and after an examination of this work and of that which is endeavoring to hold the position of a rival. The scholars of America and of England, with scarcely an exception, have decided in favor of Worcester. Not a single scholar, equal in authority to any one mentioned below, can be cited as giving, after a comparison of the two works, the preference to Webster's Dictionary. We give the testimony:—

From C. C. ELTON, LL.D., *President of Harvard College.*

Aware of the labor and care which had been devoted to this (the department of scientific terms) as well as to other parts of the work, I felt assured that Worcester's Quarto Dictionary would more nearly meet the public wants than any other hitherto published.

My expectations have been more than fulfilled. I find it not only rich beyond example in its vocabulary, but carefully elaborate in all the details, and thoroughly trustworthy as a guide to the most correct and elegant usage of the language.

From the REV. JOSEPH BOSWORTH, D.D., F.R.S., *Professor of Anglo-Saxon, Oxford, England.*

It is the most complete and practical, the very best, as well as the cheapest English Dictionary that I know.

From GEORGE P. MARSH, LL.D.

The work of Dr. Worcester is unquestionably much superior to any other general dictionary of the language in every one of those particulars (orthography, pronunciation, definition, fullness of vocabulary, and precision and distinctness of definition).

From REV. W. WHEWELL, D.D., *Master of Trinity College, England.*

I have repeatedly consulted the Dictionary since it has been in my possession, and have seen reason to think it more complete and exact than its predecessors.

From CHARLES RICHARDSON, LL.D., *the oldest living English Lexicographer, England.*

I sincerely hope you may enjoy from your brethren, both in America and England, that tribute of honor to which you have earned so undoubted a title.

From D. R. GOODWIN, D.D., *President of Trinity College, Hartford.*

It was but a short time since that I was led to commend another dictionary as, on the whole, and with some exceptions, the best and most complete thing of the kind within my knowledge. The commendation was honestly given at the time; but now it must be withdrawn in favor of yours. I consider your dictionary, in orthography, pronunciation, and definitions, as superior to any of its predecessors.

From REV. W. B. SPRAGUE, D.D., *of Albany, N. Y.*

My opinion of Worcester's Quarto Dictionary, after having given it as extended an examination as my circumstances would admit, is, that there is no other dictionary in the language that compares with it for completeness, accuracy, comprehensiveness, and precision, and perhaps I ought to add, that I have arrived at this conclusion rather contrary to a preconceived opinion.

From REV. HENRY A. BOARDMAN, D.D., *of Philadelphia.*

I particularly like it (the Dictionary), 1. Because of its very comprehensive character: 2. Because it adheres to the settled orthography of our noble language,—discarding those innovations which, however countenanced by certain publishing-houses, have never to any extent been accepted by the scholars of our country.

From LOUIS AGASSIZ, LL.D.

It is of great importance, when the nomenclature of science is gradually creeping into common use, that an English lexicon should embrace as much of it as is consistent with the language we speak. I am truly surprised and highly delighted to find you have succeeded far beyond my expectations in making the proper selection, and combining with it a remarkable degree of accuracy. More could hardly be given except in a scientific cyclopædia.

The following lines are quoted from Harper's Magazine for September. They serve to show very truthfully the comparative value of recent and old commendations:—

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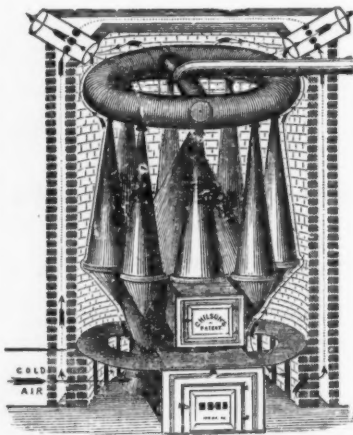
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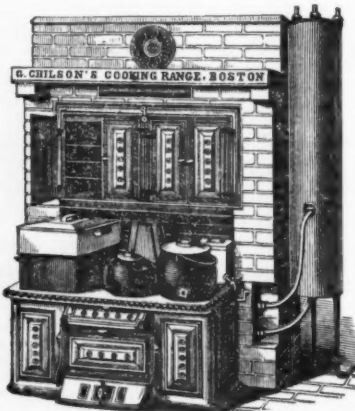
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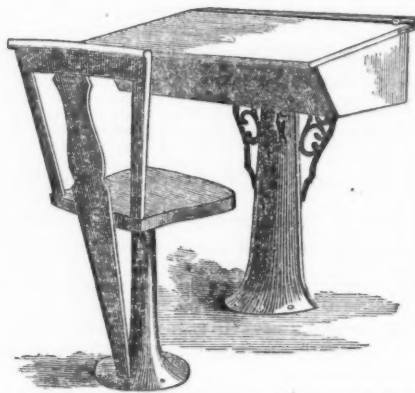
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